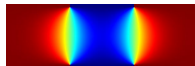


1

# Liebmann technical documentation

2



3

Laplace equation 2D (ZR)

4

(Cylindrical coordinates).

5

relaxation scheme explained.

6

(5 - point star)

7

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10

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**version 9**

12

**2024.06.09**

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Lublin, Poland

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## 118 1 Liebmann technical documentation series

- 119 1. Wyznaczanie rozkładu pola elektrostatycznego w próżni metodą relak-  
120 sacyjną Liebmann. (Polish version / wersja polska)
- 121 2. Determination of electrostatic field distribution by using Liebmann relax-  
122 ation method. (English version / wersja angielska)
- 123 3. Graphics. Mapping voltages to colours. (colormaps)
- 124 4. Laplace equation 2D (XY). (Cartesian coordinates). Relaxation scheme  
125 explained. (5 - point star)
- 126 5. Laplace equation 2D (ZR) (Cylindrical coordinates). Relaxation scheme  
127 explained. (5 - point star)
- 128 6. Liebmann source code. (ANSI C programming language)

## 129 2 Versions of this document

- 130 1. version 1 - 2023.11.03
- 131 2. version 2 - 2023.01.04
- 132 3. version 3 - 2024.02.02
- 133 4. version 4 - 2024.04.02
- 134 5. version 5 - 2024.05.18
- 135 6. version 6 - 2024.05.23
- 136 7. version 7 - 2024.05.24
- 137 8. version 8 - 2024.06.06 (complete  $P_1..P_9$ )
- 138 9. version 9 - 2024.06.09

## 139 3 Solving Laplace equation using relaxation method

140 I tried to solve Laplace equation using mainly information from Pierre Grivet's  
141 book (Electron Optics) - [1].  
142 There are few editions of this book (1965, 1972). Second edition (1972) con-  
143 tains explanation of relaxation method (page 38).  
144 More generalized approaches has been drafted by James R. Nagel - [2].  
145 <https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/> (visited 2023-03-01).  
146

147 Taylor expansion in cylindrical coordinates has been found on the Internet:  
148 [3].

149  
150 There are also publications edited by Albert Septier: Focusing of Charged  
151 Particles [4] and Applied Charged Particle Optics (part A). [5].

152 I have also found some ideas in publication of D W O Heddle: Electrostatic  
153 Lens Systems [6] (especially using PC computers to solve electrostatic prob-  
154 lems).

155 I have also found (brief) description of by - hand solving of Laplace equa-  
156 tion by Bohdan Paszkowski - [7] (Polish edition). English translation of this book  
157 also exists - [8].

158  
159 I would like to thank many people, who helped me with this challenge. Espe-  
160 cially prof. dr hab. Mieczysław Jałochowski (supervisor of my master's thesis),  
161 who enabled me to use SIMION and MATLAB software while writing master's  
162 thesis about electron optical systems at University of Maria Curie - Skłodowska  
163 in Lublin in 2008. I would also thank to prof. Marcin Turek for fruitful discus-  
164 sion about numerical methods. What is more, my colleague Bartosz in 2012  
165 had explained me general problems with software efficiency. So he had also  
166 contributed significantly to the idea of Liebmann software (especially using C  
167 language).

## 168 4 Explanation of symbols in calculations

- 169 •  $P_i$  -  $i$ -th mesh node
- 170 •  $V_i$  - value of electrostatic potential at node  $P_i$ . Unit - [V]
- 171 •  $h$  - mesh step (for example  $h_x$  - mesh step in  $x$  direction). Unit - [mm]
- 172 •  $g_{i+/-}$  - gradient in direction  $i$  (for example  $g_{1z-} = \frac{V_1 - V_{1z-}}{h_z}$ ). Unit -  $\left[\frac{V}{mm}\right]$
- 173 •  $i_{row}$  - index of row in mesh. Values of  $i_{row} = 1, 2, \dots, \text{size\_row}$
- 174 •  $i_{col}$  - index of column in mesh. Values of  $i_{col} = 1, 2, \dots, \text{size\_col}$
- 175 •  $p$  - in book: - [1]  $r = ph_r$ , so for off - axis point we have:  $p = (i_{row} - 1)$

176 **5 Mesh ZR - type A (on axis)**

177  $h_z \neq h_r$

178 gradient  $V$  outside a mesh exists

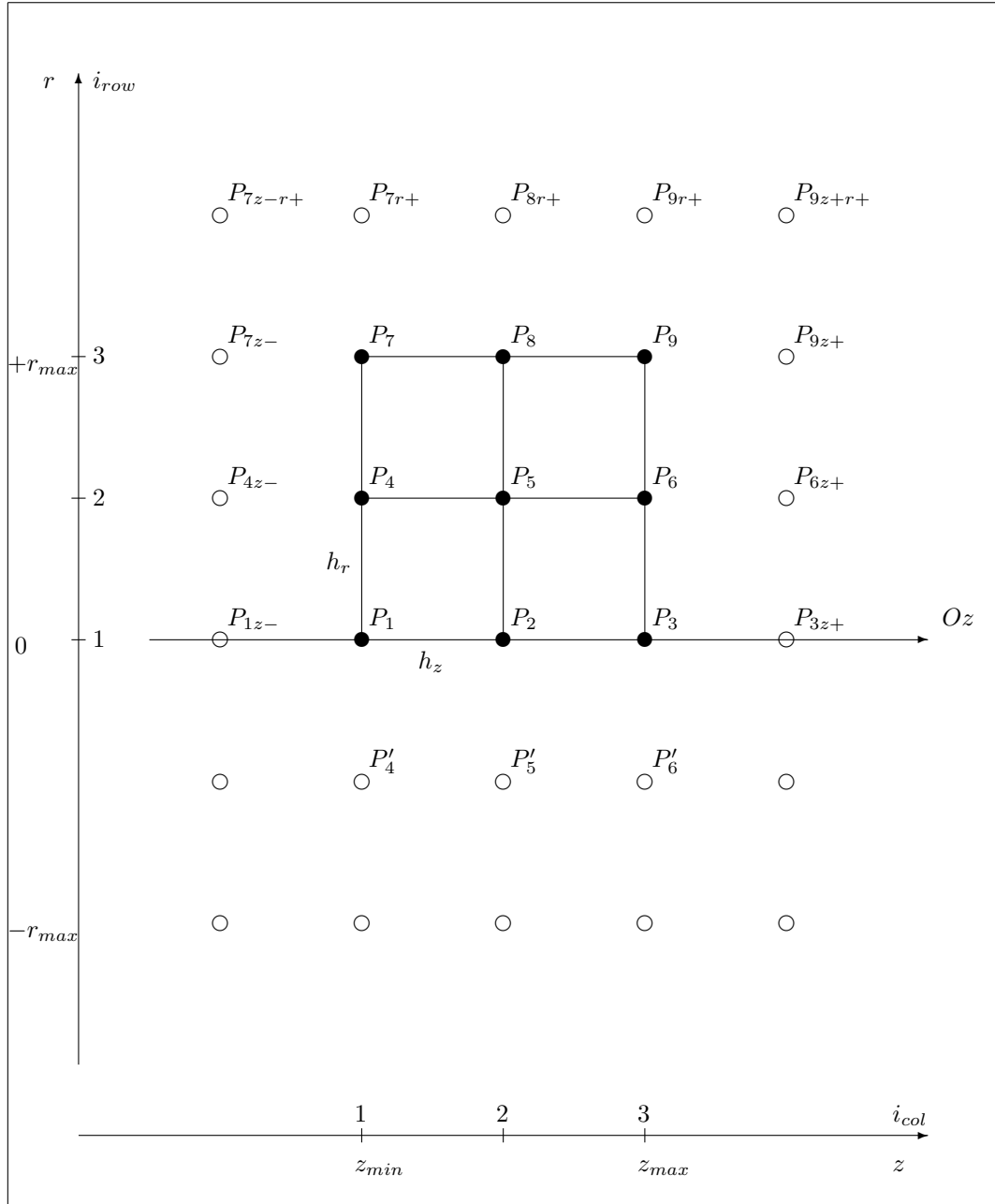


Figure 1: Mesh ZR type A



179 **6 Mesh ZR - type B (on axis)**

180  $h_z \neq h_r$

181 gradient  $V$  outside a mesh does not exist

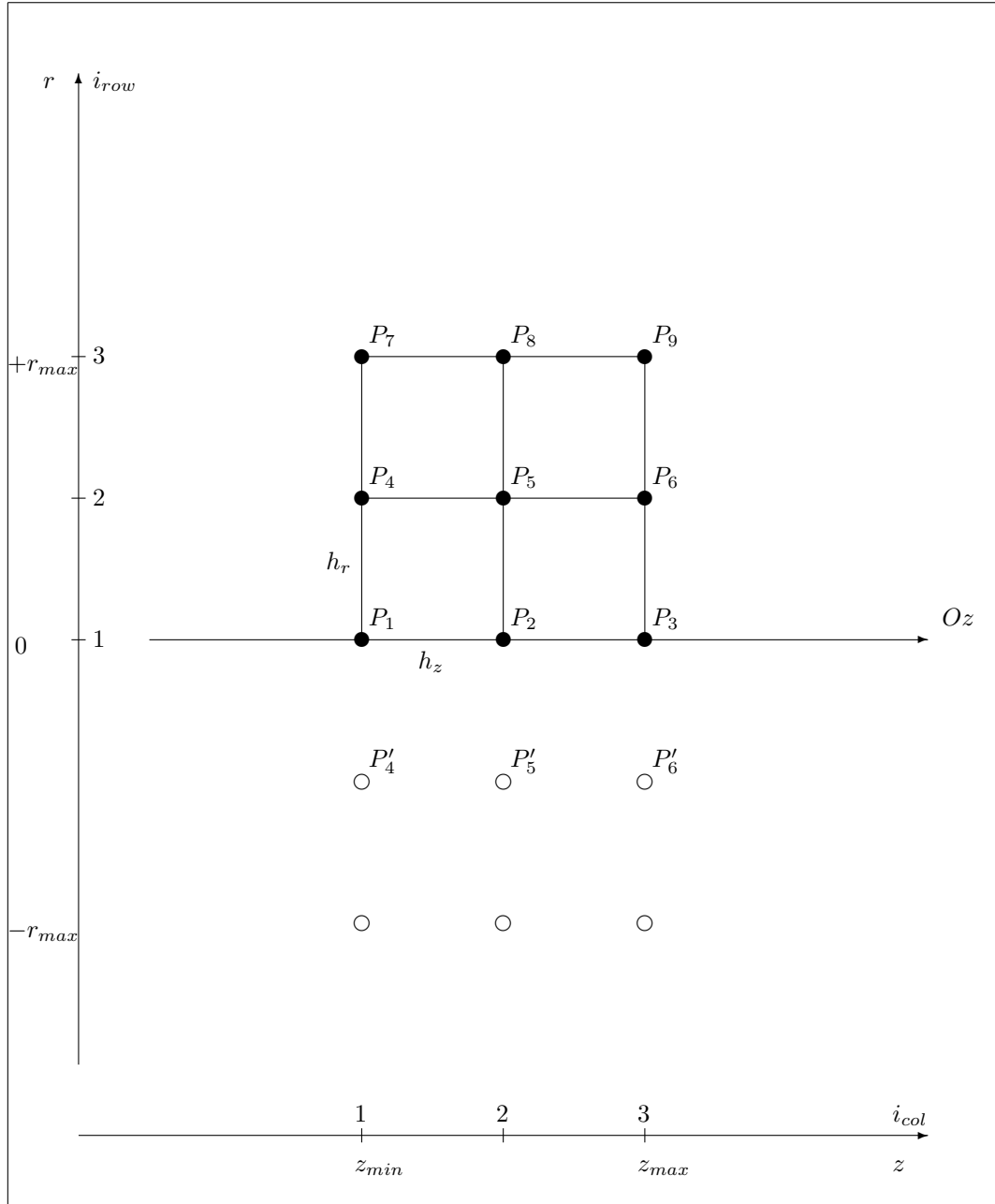


Figure 2: Mesh ZR type B

182 **7 Mesh ZR - type C (on axis)**

183  $h_z = h_r = h$

184 gradient  $V$  outside a mesh exists

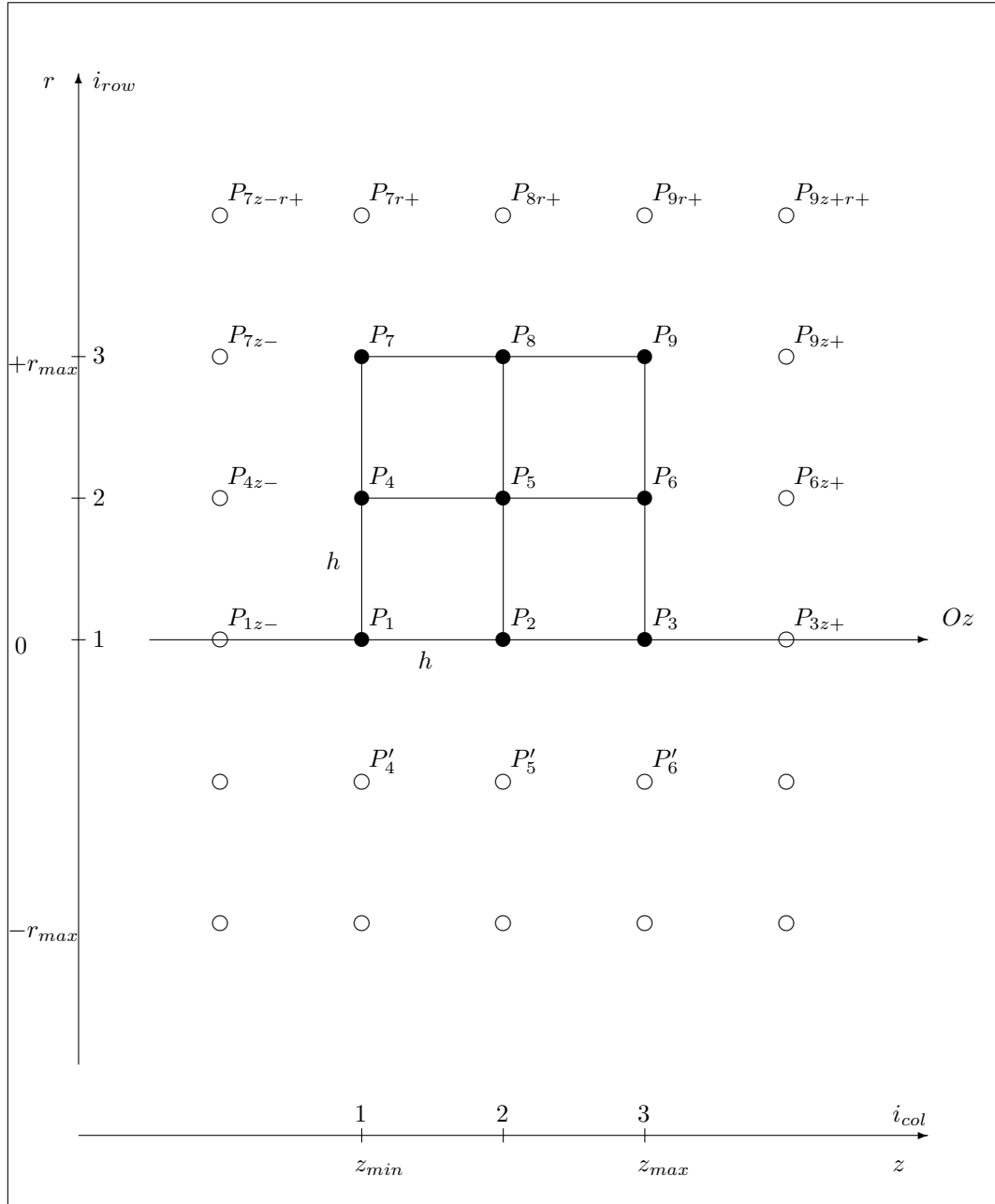


Figure 3: Mesh ZR type C

185 **8 Mesh ZR - type D (on axis)**

186  $h_z = h_r = h$

187 gradient  $V$  outside a mesh does not exist

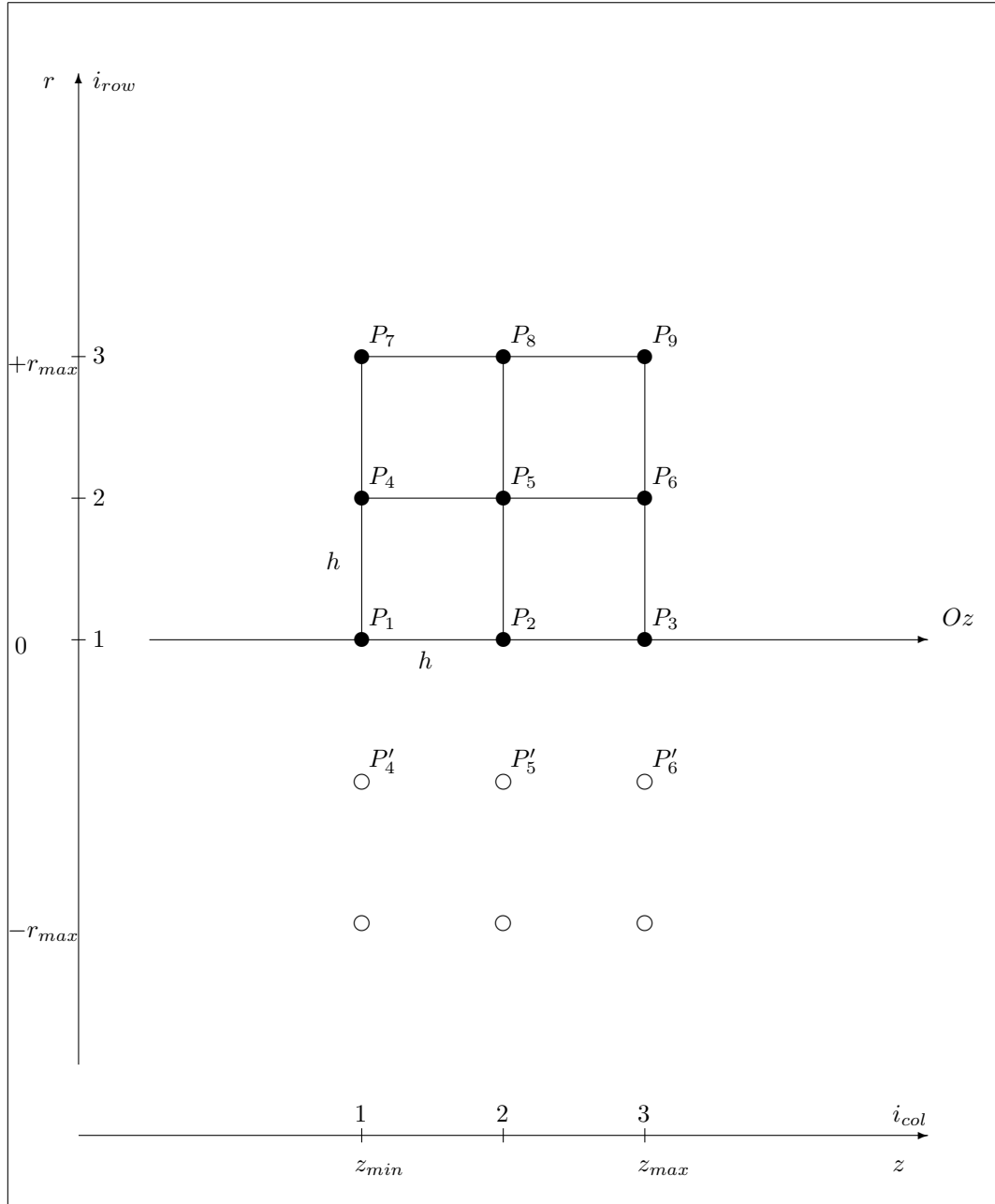


Figure 4: Mesh ZR type D

## 188 9 Example of A-type mesh in ANSI C (on axis)

189 Example of A- type mesh in ANSI C program. The mesh is represented by 2  
 190 dimensional array of double precision numbers. Rows and columns in mesh  
 191 are numbered from 1 (this was my choice) instead of default 0 (as usual in C  
 192 language). This choice has pros and cons. Is is easier to calculate mesh size  
 193 (size\_row \* size\_col). Access to each node can be also more intuitive, but logic  
 194 in each library function must contain this shift between node ordering styles.

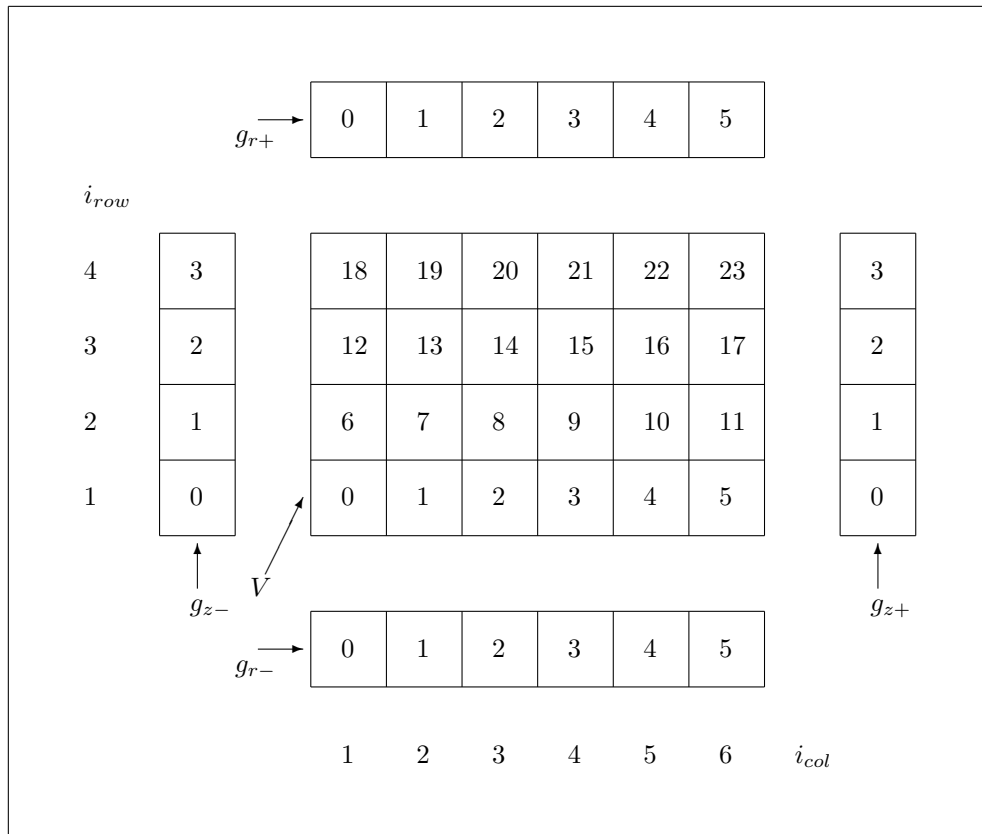


Figure 5: ANSI C - mesh XY type A

195 Note. This is more general example of „off-axis” mesh. If bottom egde of  
 196 mesh lies on axis  $Oz$ , then gradient  $g_{r-}$  does not exist.

- 197 •  $g_{z-} \equiv \text{double* ptr\_gZ\_minus}$
- 198 •  $g_{z+} \equiv \text{double* ptr\_gZ\_plus}$
- 199 •  $g_{r-} \equiv \text{double* ptr\_gR\_minus}$
- 200 •  $g_{r+} \equiv \text{double* ptr\_gR\_plus}$

```

201     •  $V \equiv \text{double}^* \text{ptr}_V$ 
202     • unsigned int size_row == 4
203     • unsigned int size_col == 6
204     • unsigned int i_row == 1, 2, ..., 4
205     • unsigned int i_col == 1, 2, ..., 6
206     • double h_z == 1.0 [mm]
207     • double h_r == 2.0 [mm]

```

208 The following picture describes analogous version of `ptr_V` mesh, which  
 209 can be dynamically allocated on heap by pointer metod. The mesh is rep-  
 210 resented by single block of memory. The numbers or rows and columns are  
 211 also known, so each node can be also accessed by appropriate index (memory  
 212 address).

213 The following picture describes analogous version of mesh, which can be  
 214 easily dynamically allocated on heap by pointer metod. The mesh is repre-  
 215 sented by single block of memory. The numbers or rows and columns are also  
 216 known, so each node can be also accessed by appropriate index (memory ad-  
 217 dress).

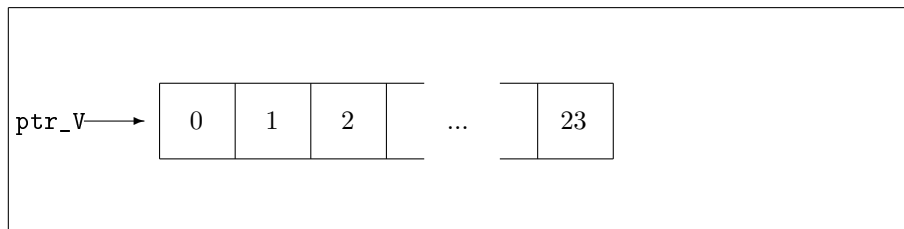


Figure 6: ANSI C - mesh ZR type D

218 Each mesh point has its unique index (let's say `icp` - (index of central  
 219 point)), which can be determined, if we know indices of row and column (`i_row`,  
 220 `i_col`).

$$\text{icp} == (\text{i\_row} - 1) * \text{size\_col} + \text{i\_col} - 1 \quad (9.1)$$

221 For example for each point of a mesh indices of row and column have val-  
 222 ues:

$$\begin{aligned} \text{i\_row} &== 1, 2, \dots, \text{size\_row} \\ \text{i\_col} &== 1, 2, \dots, \text{size\_col} \end{aligned} \quad (9.2)$$

223        Also for any relaxation formula for off - axis case the  $p$  symbol appears. This  
224        symbol is connected with  $r$  cylindrical coordinate of given node:  
225

$$r = ph_r \quad (9.3)$$

226        SO:

$$p == (i\_row - 1) \quad (9.4)$$

## 227 10 Example of B-type mesh in ANSI C (on axis)

228 Example of B- type mesh in ANSI C program. The mesh is analogous to A -  
229 type mesh. There are no electric field gradients on mesh borders.

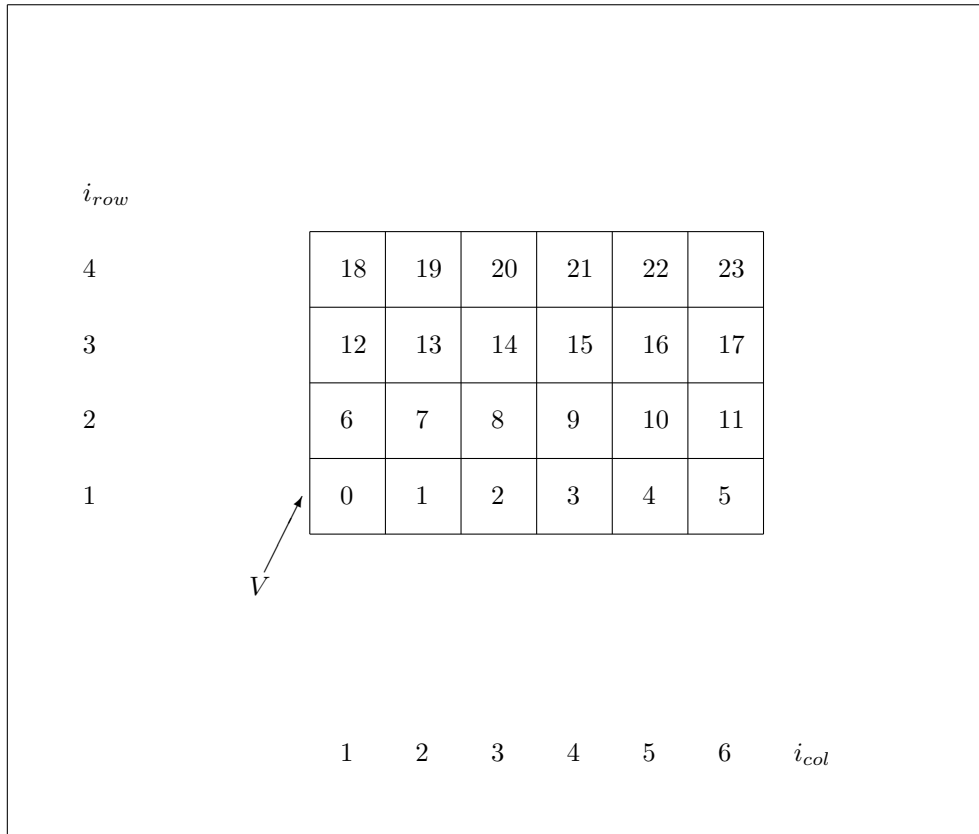


Figure 7: ANSI C - mesh XY type B

```

230 •  $V \equiv \text{double* ptr}_V$ 
231 • unsigned int size_row == 4
232 • unsigned int size_col == 6
233 • unsigned int i_row == 1, 2, ..., 4
234 • unsigned int i_col == 1, 2, ..., 6
235 • double h_z == 1.0 [mm]
236 • double h_r == 2.0 [mm]

```

## 237 11 Example of C-type mesh in ANSI C (on axis)

238 Example of C- type mesh in ANSI C program. The mesh is analogous to A -  
239 type mesh. Just mesh mesh step  $h_x = h_y = h$ .

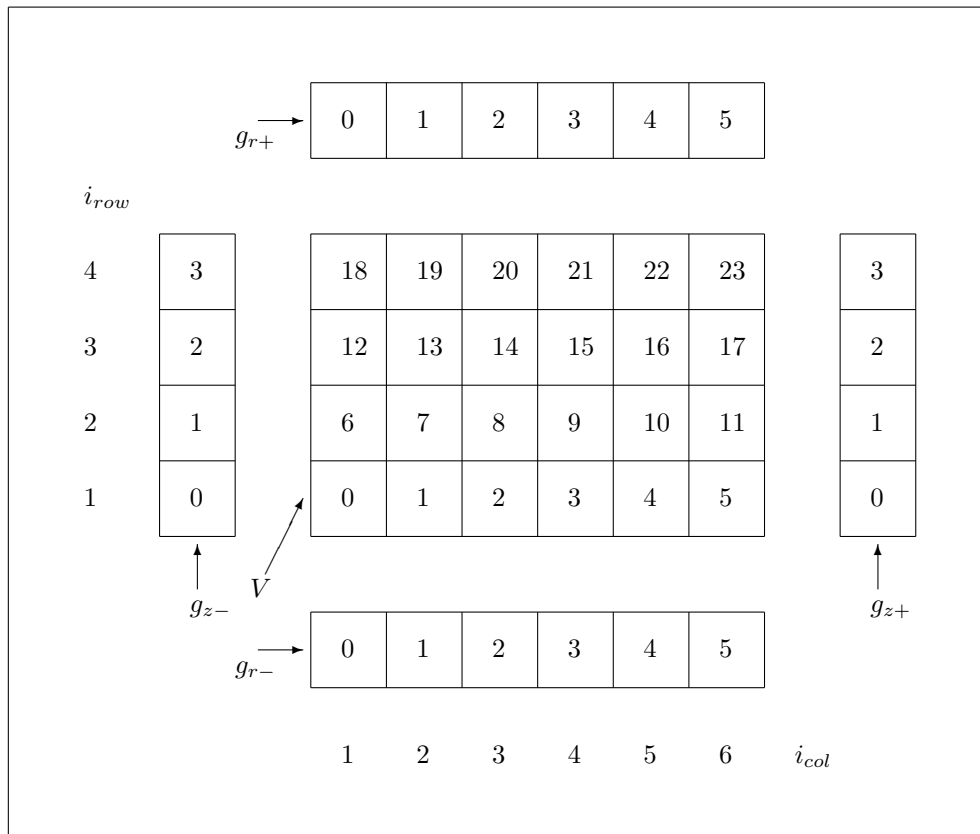


Figure 8: ANSI C - mesh XY type C

240 Note. This is more general example of „off-axis” mesh. If bottom egde of  
241 mesh lies on axis  $Oz$ , then gradient  $g_{r-}$  does not exist.

- 242 •  $g_{z-} \equiv \text{double* ptr\_gZ\_minus}$
- 243 •  $g_{z+} \equiv \text{double* ptr\_gZ\_plus}$
- 244 •  $g_{r-} \equiv \text{double* ptr\_gR\_minus}$
- 245 •  $g_{r+} \equiv \text{double* ptr\_gR\_plus}$
- 246 •  $V \equiv \text{double* ptr\_V}$
- 247 • `unsigned int size_row == 4`



```
248 • unsigned int size_col == 6
249 • unsigned int i_row == 1, 2, .., 4
250 • unsigned int i_col == 1,2, .., 6
251 • double h == 1.0 [mm]
```

## 252 12 Example of D-type mesh in ANSI C (on axis)

253 Example of D- type mesh in ANSI C program. The mesh is analogous to B -  
 254 type mesh. Just  $h_x = h_y = h$ .

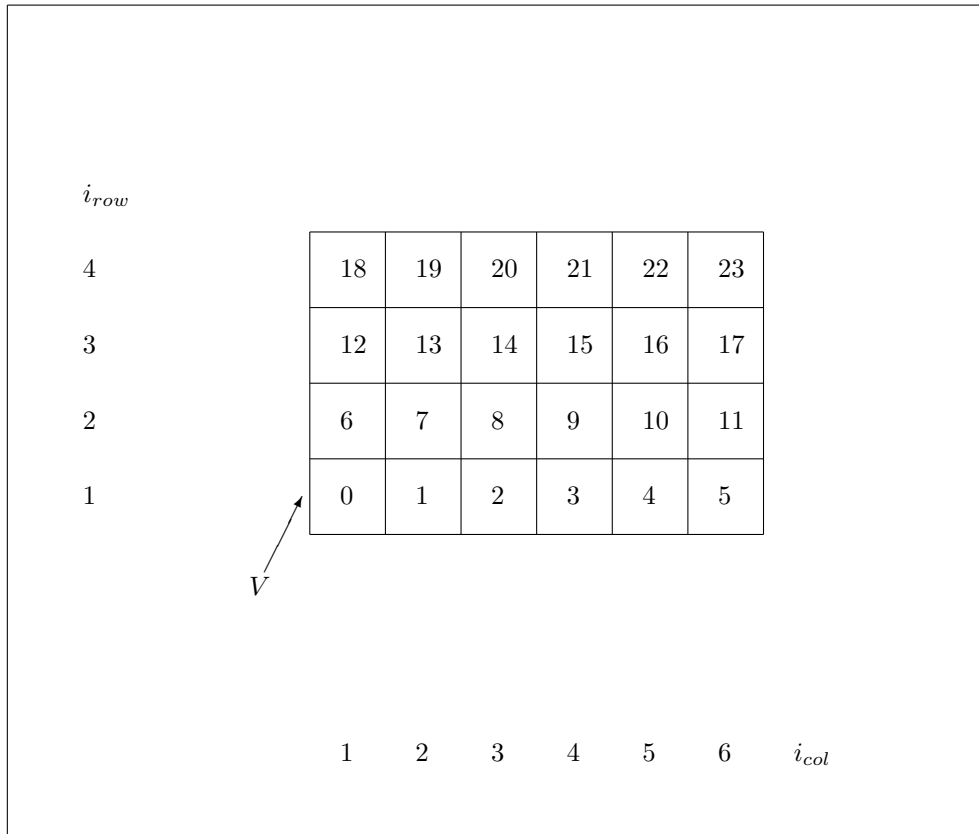


Figure 9: ANSI C - mesh ZR type D

- 255 •  $V \equiv \text{double* ptr\_V}$
- 256 • `unsigned int size_row == 4`
- 257 • `unsigned int size_col == 6`
- 258 • `unsigned int i_row == 1, 2, ..., 4`
- 259 • `unsigned int i_col == 1, 2, ..., 6`
- 260 • `double h == 1.0 [mm]`

## 261 13 Partial derivatives on $Oz$ axis

### 262 13.1 Personal note

263 This is my personal interpretation. I cannot guarantee correctness of this ap-  
264 proach

### 265 13.2 Nodes numbering (on axis $Oz$ )

266 We will try to work with  $P_2$  point (determine approximations of partial derivatives  
267 for point  $P_2$ , which lies on axis  $Oz$ ). Nodes numbering on axis  $Oz$  differs from  
268 numbering convention in Pierre Grivet's book.

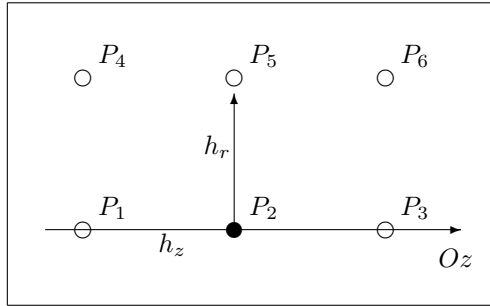


Figure 10: Nodes on axis  $Oz$

269 Point  $P_2$  is situated on  $Oz$  axis. It has 2 neighbours on axis  $Oz$  :  $P_1$  and  $P_3$ .  
270 Node  $P_5$  lies above  $P_2$  node. The mesh step in  $r$  direction is  $h_r$ . The mesh  
271 step in  $z$  direction is  $h_z$ .

### 272 13.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates

273 Taylor expansion of  $V_{(z,r)}$  function in cylindrical coordinates [3]:

$$\begin{aligned}
 V_{(z,r)} = V_{(z_0,r_0)} &+ \left( \frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
 &\left( \frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
 &\frac{1}{2!} \left( \frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
 \end{aligned} \tag{13.1}$$

### 274 13.4 Laplace operator in rotationally symmetrical systems

275 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3  
276 elements [1] (on page 42):

$$\nabla^2 (V_{(z,r)}) = \left( \frac{\partial^2 V}{\partial r^2} \right) + \frac{1}{r} \left( \frac{\partial V}{\partial r} \right) + \left( \frac{\partial^2 V}{\partial z^2} \right) \quad (13.2)$$

277 In this chapter we will try to determine approximation of each term.

### 278 13.5 Value of first partial derivative of $V$ with respect to $r$ on axis 279 $Oz$

280 In cylindrically symmetrical field first partial derivative of  $V$  (by  $r$ ) on axis  $Oz$   
281 equals zero (because  $V_{(+dr)} = V_{(-dr)}$ )

$$\left( \frac{\partial V}{\partial r} \right)_{(z,r=0)} = 0 \quad (13.3)$$

### 282 13.6 Value of second partial derivative of $V$ with respect to $r$ on 283 axis $Oz$

284 In this subchapter we will try to determine the first term of equation 13.2

285 In our case there is node  $P_2$  on axis  $Oz$ . The nearest neighbour of  $P_2$  is  
286 node  $P_5$ , which lies „over  $Oz$  axis“. The distance between  $P_2$  and  $P_5$  is  $h_r$ .  
287 When we „walk away“ axis  $Oz$  in  $r$  direction (from point  $P_2$  to point  $P_5$ ), the  
288 electric potential  $V_5$  can be determined from truncated Taylor expansion 13.1  
289 by expression:

$$V_5 \approx V_2 + \left( \frac{\partial V}{\partial r} \right)_{P_2} \cdot h_r + \frac{1}{2!} \left( \frac{\partial^2 V}{\partial r^2} \right)_{P_2} \cdot h_r^2 \quad (13.4)$$

290 We want to determine the second derivative:

$$\left( \frac{\partial^2 V}{\partial r^2} \right)_{P_2} = ? \quad (13.5)$$

291 We solve equation 13.4 (using relation 13.3).

$$\left( \frac{\partial^2 V}{\partial r^2} \right)_{P_2} \approx \frac{2! (V_5 - V_2)}{h_r^2} = \frac{2 (V_5 - V_2)}{h_r^2} \quad (13.6)$$

292 This is final form of approximation of the second derivative of  $V$  with respect  
293 to  $r$  on axis  $Oz$ . It will help us to determine Laplace operator in rotationally  
294 symmetrical systems.

295 **13.7 Value of first partial derivative of  $V$  with respect to  $r$  divided**  
 296 **by  $r$  on axis  $Oz$**

297 We will try to determine the second term of relation 13.2 When we are on  $Oz$   
 298 axis, the second term has to be determined (because it aims to value  $\frac{0}{0}$ ).

299 When we „walk away” axis  $Oz$  in  $r$  direction, the electric potential  $V_{(z_0,r)}$   
 300 can be determined from truncated Taylor expansion by:

301

$$V_{(z_0,r)} \approx V_{(z_0,r_0)} + \left( \frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) \quad (13.7)$$

302 On  $Oz$  axis  $r_0 = 0$ , so  $(r_0 - r) = r$

303

304 Thus we have:

$$V_{(z_0,r)} \approx V_{(z_0,0)} + \left( \frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot r \quad (13.8)$$

305 Now let us differentiate (both sides) of such relation:

$$\left| \frac{\partial}{\partial r} \right. \quad (13.9)$$

306 We get:

$$\left( \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left( \frac{\partial V}{\partial r} \right)_{(z_0,0)} + \left( \frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r + \left( \frac{\partial V}{\partial r} \right)_{(z_0,0)} \cdot 1 \quad (13.10)$$

307 On axis  $Oz$  we can apply relation 13.3. That's why we can remove these  
 308 two terms (first and third) from equation 13.10:

309 So we get (if  $r = 0$ ):

$$\left( \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left( \frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \cdot r \quad (13.11)$$

310 We can now divide both sides by  $r$ .

$$\left| \cdot \frac{1}{r} \right. \quad (13.12)$$

311 We have relation, which has been published in Pierre Grivet's book[1].

$$\left( \frac{1}{r} \frac{\partial V}{\partial r} \right)_{(z_0,r)} \approx \left( \frac{\partial^2 V}{\partial r^2} \right)_{(z_0,0)} \quad (13.13)$$

312 Approximation of this term on numerical mesh has been already determined  
 313 in previous subsection (13.6).

314 **13.8 Value of second partial derivative of  $V$  with respect to  $z$  on**  
 315 **axis  $Oz$**

316 The third term of Laplace operator in rotationally symmetrical systems 13.2  
 317 takes form ( on picture 10 ):

$$\left( \frac{\partial^2 V}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (13.14)$$

318 Now we have determined all the 3 approximations o partial derivatives of  $V$   
 319 in cylindrically symmetrical systems (on axis  $O_z$ ).

## 320 14 Partial derivatives off $Oz$ axis

### 321 14.1 Personal note

322 This is my presonal interpretation. I cannot guarantee correctness of this ap-  
323 proach

### 324 14.2 Nodes numbering in Liebmann mesh (off axis $O_z$ )

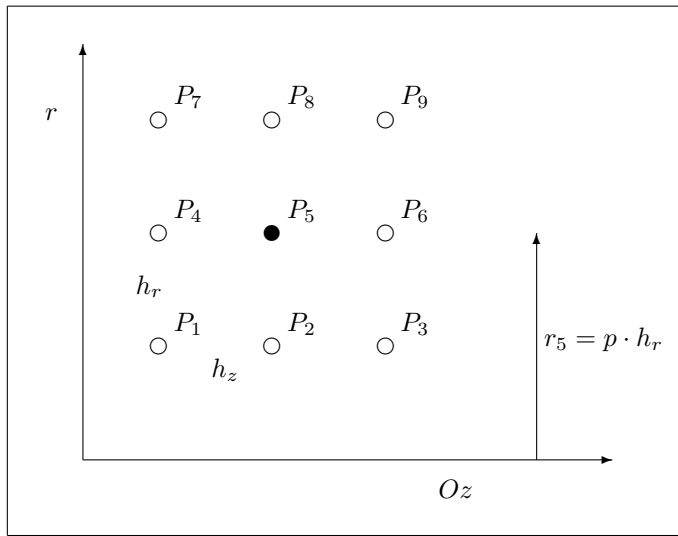


Figure 11: Nodes off axis  $O_z$ . Exemplary vector  $r_5$  describes distance from axis  $O_z$  to node  $P_5$

325 Mesh step in  $z$  direction is  $h_z$ . Mesh step in  $r$  direction is  $h_r$ . Sample mesh  
326 points  $P_5$  lies off  $O_z$  axis. Distance between mesh point  $P_5$  and  $O_z$  axis is  $r_5$ .

327 For ANSI C meshes (in Liebmann source code) the following relations have  
328 place:

$$r = p h_r \quad (14.1)$$

$$p = i_{row} - 1 \quad (14.2)$$

### 329 14.3 Taylor expansion of $V_{(z,r)}$ function in cylindrical coordinates

330 Taylor expansion of  $V_{(z,r)}$  function in cylindrical coordinates [3]:

$$\begin{aligned}
V_{(z,r)} = V_{(z_0,r_0)} + \left( \frac{\partial V}{\partial z} \right)_{(z_0,r_0)} (z - z_0) + \\
\left( \frac{\partial V}{\partial r} \right)_{(z_0,r_0)} (r - r_0) + \\
\frac{1}{2!} \left( \frac{\partial^2 V}{\partial z^2} \right)_{(z_0,r_0)} (z - z_0)^2 + \dots
\end{aligned} \tag{14.3}$$

#### 331 14.4 Laplace operator in rotationally symmetrical systems

332 Laplace operator in cylindrical coordinates (cylindrical symmetry) consist of 3  
333 elements [1] (on page 42):

$$\begin{aligned}
\nabla^2 (V_{(z,r)}) = \left( \frac{\partial^2 V}{\partial r^2} \right) + \\
\frac{1}{r} \left( \frac{\partial V}{\partial r} \right) + \\
\left( \frac{\partial^2 V}{\partial z^2} \right)
\end{aligned} \tag{14.4}$$

334 In this chapter we will try to determine approximation of each term.

#### 335 14.5 Value of second partial derivative of $V$ with respect to $r$ off 336 axis $Oz$

$$\left( \frac{\partial^2 V}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \tag{14.5}$$

#### 337 14.6 Value of first partial derivative of $V$ with respect to $r$ divided 338 by $r$ off axis $Oz$

$$\frac{1}{r_5} \left( \frac{\partial V}{\partial r} \right)_{P_5} \approx \frac{1}{r_5} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2r_5 h_z} \tag{14.6}$$

#### 339 14.7 Value of second partial derivative of $V$ with respect to $z$ off 340 axis $Oz$

$$\left( \frac{\partial^2 V}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \tag{14.7}$$



## 341 15 Relaxation formula for node P1 (on axis $Oz$ )

### 342 15.1 Node description

343 Left, bottom corner of mesh ZR (on axis  $Oz$ ).

### 344 15.2 Calculation of relaxation formula

345 Laplace equation at node  $P_1$

$$\nabla^2 (V_{(z,r)})_{P_1} = 0 \quad (15.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} = 0 \quad (15.2)$$

346 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_1$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_1} \approx \frac{\frac{V_4 - V_1}{h_r} - \frac{V_1 - V_4}{h_r}}{h_r} = \frac{2(V_4 - V_1)}{h_r^2} \quad (15.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_1} \approx \frac{2(V_4 - V_1)}{h_r^2} \quad (15.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_1} \approx \frac{\frac{V_2 - V_1}{h_z} - \frac{V_1 - V_{1z-}}{h_z}}{h_z} = \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} \quad (15.5)$$

347 Let us substitute approximations to Laplace equation.

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} - \frac{g_{1z-}}{h_z} = 0 \quad (15.6)$$

$$\frac{2(V_4 - V_1)}{h_r^2} + \frac{2(V_4 - V_1)}{h_r^2} + \frac{V_2 - V_1}{h_z^2} = \frac{g_{1z-}}{h_z} \quad (15.7)$$

348 Let us find  $V_1$

$$V_1 = ? \quad (15.8)$$

349 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (15.9)$$

350 We obtain

$$2V_4 h_z^2 - 2V_1 h_z^2 + 2V_4 h_z^2 - 2V_1 h_z^2 + V_2 h_r^2 - V_1 h_r^2 = g_{1z-} h_z h_r^2 \quad (15.10)$$

351 Let us simplify this equation:

$$V_1 (2h_z^2 + 2h_z^2 + h_r^2) = 2V_4h_z^2 + 2V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.11)$$

352 So we have:

$$V_1 (4h_z^2 + h_r^2) = 4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2 \quad (15.12)$$

### 353 15.3 Final forms of relaxation formula

#### 354 15.3.1 zrLV\_RELAX5\_P1\_ON\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2 - g_{1z-}h_zh_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.13)$$

#### 355 15.3.2 zrLV\_RELAX5\_P1\_ON\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4h_z^2 + V_2h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (15.14)$$

#### 356 15.3.3 zrLV\_RELAX5\_P1\_ON\_C

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &\neq 0 \\ V_1 &= \frac{4V_4 + V_2 - g_{1z-}h}{5} \end{aligned} \quad (15.15)$$

#### 357 15.3.4 zrLV\_RELAX5\_P1\_ON\_D

$$\begin{aligned} h_z &= h_r = h \\ g_{1z-} &= 0 \\ V_1 &= \frac{4V_4 + V_2}{5} \end{aligned} \quad (15.16)$$

## 358 **16 Relaxation formula for node P2 (on axis $Oz$ )**

### 359 **16.1 Node description**

360 Bottom edge of mesh ZR (on axis  $Oz$ ).

### 361 **16.2 Calculation of relaxation formula**

362 Laplace equation at node  $P_2$

$$\nabla^2 (V_{(z,r)})_{P_2} = 0 \quad (16.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} = 0 \quad (16.2)$$

363 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_2$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_2} \approx \frac{\frac{V_5 - V_2}{h_r} - \frac{V_2 - V_5}{h_r}}{h_r} = \frac{2(V_5 - V_2)}{h_r^2} \quad (16.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_2} \approx \frac{2(V_5 - V_2)}{h_r^2} \quad (16.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_2} \approx \frac{\frac{V_3 - V_2}{h_z} - \frac{V_2 - V_1}{h_z}}{h_z} = \frac{V_1 + V_3 - 2V_2}{h_z^2} \quad (16.5)$$

364 Let us substitute approximations to Laplace equation.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.6)$$

365 There are no  $g$  expressions to move, to formula 7 has identical form as  
366 formula 6.

$$\frac{2(V_5 - V_2)}{h_r^2} + \frac{2(V_5 - V_2)}{h_r^2} + \frac{V_1 + V_3 - 2V_2}{h_z^2} = 0 \quad (16.7)$$

367 Let us find  $V_2$

$$V_2 = ? \quad (16.8)$$

368 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (16.9)$$

369 We obtain

$$2V_5 h_z^2 - 2V_2 h_z^2 + 2V_5 h_z^2 - 2V_2 h_z^2 + V_1 h_r^2 + V_3 h_r^2 - 2V_2 h_r^2 = 0 \quad (16.10)$$

370 Let us simplify this equation:

$$V_2 (2h_z^2 + 2h_z^2 + 2h_r^2) = 2V_5h_z^2 + 2V_5h_z^2 + V_1h_r^2 + V_3h_r^2 \quad (16.11)$$

371 So we have:

$$V_2 (4h_z^2 + 2h_r^2) = 4V_5h_z^2 + (V_1 + V_3)h_r^2 \quad (16.12)$$

### 372 **16.3 Final forms of relaxation formula**

#### 373 **16.3.1 zrLV\_RELAX5\_P2\_ON\_A**

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5h_z^2 + (V_1 + V_3)h_r^2}{4h_z^2 + 2h_r^2} \quad (16.13)$$

#### 374 **16.3.2 zrLV\_RELAX5\_P2\_ON\_B**

$$h_z \neq h_r$$

$$V_2 = \frac{4V_5h_z^2 + (V_1 + V_3)h_r^2}{4h_z^2 + 2h_r^2} \quad (16.14)$$

#### 375 **16.3.3 zrLV\_RELAX5\_P2\_ON\_C**

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.15)$$

#### 376 **16.3.4 zrLV\_RELAX5\_P2\_ON\_D**

$$h_z = h_r = h$$

$$V_2 = \frac{4V_5 + V_1 + V_3}{6} \quad (16.16)$$

## 377 17 Relaxation formula for node P3 (on axis Oz)

### 378 17.1 Node description

379 Right, bottom corner of mesh ZR (on axis Oz).

### 380 17.2 Calculation of relaxation formula

381 Laplace equation at node  $P_3$

$$\nabla^2 (V_{(z,r)})_{P_3} = 0 \quad (17.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} = 0 \quad (17.2)$$

382 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_3$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_3} \approx \frac{\frac{V_6 - V_3}{h_r} - \frac{V_3 - V_6}{h_r}}{h_r} = \frac{2(V_6 - V_3)}{h_r^2} \quad (17.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_3} \approx \frac{2(V_6 - V_3)}{h_r^2} \quad (17.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_3} \approx \frac{\frac{V_{3z+} - V_3}{h_z} - \frac{V_3 - V_2}{h_z}}{h_z} = \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} \quad (17.5)$$

383 Let us substitute approximations to Laplace equation.

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} + \frac{g_{3z+}}{h_z} = 0 \quad (17.6)$$

$$\frac{2(V_6 - V_3)}{h_r^2} + \frac{2(V_6 - V_3)}{h_r^2} + \frac{V_2 - V_3}{h_z^2} = -\frac{g_{3z+}}{h_z} \quad (17.7)$$

384 Let us find  $V_3$

$$V_3 = ? \quad (17.8)$$

385 Let us multiply both sides

$$| \cdot h_z^2 h_r^2 \quad (17.9)$$

386 We obtain

$$2V_6 h_z^2 - 2V_3 h_z^2 + 2V_6 h_z^2 - 2V_3 h_z^2 + V_2 h_r^2 - V_3 h_r^2 = -g_{3z+} h_z h_r^2 \quad (17.10)$$

387 Let us simplify this equation:

$$V_3 (2h_z^2 + 2h_r^2 + h_r^2) = 2V_6 h_z^2 + 2V_6 h_r^2 + V_2 h_r^2 + g_{3z+} h_z h_r^2 \quad (17.11)$$

388 So we have:

$$V_3 (4h_z^2 + h_r^2) = 4V_6 h_z^2 + V_2 h_r^2 + g_{1z-} h_z h_r^2 \quad (17.12)$$

### 389 17.3 Final forms of relaxation formula

#### 390 17.3.1 zrLV\_RELAX5\_P3\_ON\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{3z+} &\neq 0 \\ V_3 &= \frac{4V_6 h_z^2 + V_2 h_r^2 + g_{3z+} h_z h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.13)$$

#### 392 17.3.2 zrLV\_RELAX5\_P3\_ON\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{3z+} &= 0 \\ V_3 &= \frac{4V_6 h_z^2 + V_2 h_r^2}{4h_z^2 + h_r^2} \end{aligned} \quad (17.14)$$

#### 393 17.3.3 zrLV\_RELAX5\_P3\_ON\_C

$$\begin{aligned} h_z &= h_r = h \\ g_{3z+} &\neq 0 \\ V_3 &= \frac{4V_6 + V_2 + g_{3z+} h}{5} \end{aligned} \quad (17.15)$$

#### 394 17.3.4 zrLV\_RELAX5\_P3\_ON\_D

$$\begin{aligned} h_z &= h_r = h \\ g_{3z+} &= 0 \\ V_3 &= \frac{4V_6 + V_2}{5} \end{aligned} \quad (17.16)$$

## 395 18 Relaxation formula for node P4

### 396 18.1 Node description

397 Left edge of mesh ZR.

### 398 18.2 Calculation of relaxation formula

399 Laplace equation at node  $P_4$

$$\nabla^2 (V_{(z,r)})_{P_4} = 0 \quad (18.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} = 0 \quad (18.2)$$

400 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_4$

401

402 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_4} \approx \frac{\frac{V_7 - V_4}{h_r} - \frac{V_4 - V_1}{h_r}}{h_r} = \frac{V_1 + V_7 - 2V_4}{h_r^2} \quad (18.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_4} \approx \frac{1}{r} \frac{V_7 - V_1}{2h_r} = \frac{V_7 - V_1}{2ph_r^2} \quad (18.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_4} \approx \frac{\frac{V_5 - V_1}{h_z} - \frac{V_4 - V_{4z-}}{h_z}}{h_z} = \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} \quad (18.5)$$

403 Let us substitute approximations to Laplace equation.

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} - \frac{g_{4z-}}{h_z} = 0 \quad (18.6)$$

$$\frac{V_1 + V_7 - 2V_4}{h_r^2} + \frac{V_7 - V_1}{2ph_r^2} + \frac{V_5 - V_4}{h_z^2} = \frac{g_{4z-}}{h_z} \quad (18.7)$$

404 Let us find  $V_4$

$$V_4 = ? \quad (18.8)$$

405 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (18.9)$$

406 We obtain

$$2pV_1h_z^2 + 2pV_7h_z^2 - 4pV_4h_z^2 + V_7h_z^2 - V_1h_z^2 + 2pV_5h_r^2 - 2pV_4h_r^2 = 2pg_{4z-}h_zh_r^2 \quad (18.10)$$

407 Let us simplify this equation:

$$V_4(4ph_z^2 + 2ph_r^2) = V_1(2ph_z^2 - h_z^2) + V_7(2ph_z^2 + h_z^2) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.11)$$

408 So we have:

$$V_42p(2h_z^2 + h_r^2) = V_1h_z^2(2p-1) + V_7h_z^2(2p+1) + V_52ph_r^2 - 2pg_{4z-}h_zh_r^2 \quad (18.12)$$

### 409 18.3 Final forms of relaxation formula

#### 410 18.3.1 zrLV\_RELAX5\_P4\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{4z-} &\neq 0 \end{aligned}$$

$$V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 - \frac{g_{4z-}h_zh_r^2}{2h_z^2 + h_r^2} \quad (18.13)$$

#### 412 18.3.2 zrLV\_RELAX5\_P4\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{4z-} &= 0 \end{aligned}$$

$$V_4 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_1 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_7 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (18.14)$$

#### 414 18.3.3 zrLV\_RELAX5\_P4\_C

$$\begin{aligned} h_z &= h_r = h \\ g_{4z-} &\neq 0 \end{aligned}$$

$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 - \frac{g_{4z-}h}{3} \quad (18.15)$$

#### 416 18.3.4 zrLV\_RELAX5\_P4\_D

$$\begin{aligned} h_z &= h_r = h \\ g_{4z-} &= 0 \end{aligned}$$

$$V_4 = \frac{2p-1}{6p}V_1 + \frac{2p+1}{6p}V_7 + \frac{1}{3}V_5 \quad (18.16)$$



## 417 19 Relaxation formula for node P5

### 418 19.1 Node description

419 Inner node of mesh ZR.

### 420 19.2 Calculation of relaxation formula

421 Laplace equation at node  $P_5$

$$\nabla^2 (V_{(z,r)})_{P_5} = 0 \quad (19.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} = 0 \quad (19.2)$$

422 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_5$

423

424 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_5} \approx \frac{\frac{V_8 - V_5}{h_r} - \frac{V_5 - V_2}{h_r}}{h_r} = \frac{V_2 + V_8 - 2V_5}{h_r^2} \quad (19.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_5} \approx \frac{1}{r} \frac{V_8 - V_2}{2h_r} = \frac{V_8 - V_2}{2ph_r^2} \quad (19.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_5} \approx \frac{\frac{V_6 - V_5}{h_z} - \frac{V_5 - V_4}{h_z}}{h_z} = \frac{V_4 + V_6 - 2V_5}{h_z^2} \quad (19.5)$$

425 Let us substitute approximations to Laplace equation.

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.6)$$

426 We don't need to simplify this equation in step 7:

$$\frac{V_2 + V_8 - 2V_5}{h_r^2} + \frac{V_8 - V_2}{2ph_r^2} + \frac{V_4 + V_6 - 2V_5}{h_z^2} = 0 \quad (19.7)$$

427 Let us find  $V_5$

$$V_5 = ? \quad (19.8)$$

428 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (19.9)$$

429 We obtain

$$2pV_2h_z^2 + 2pV_8h_z^2 - 4pV_5h_z^2 + V_8h_z^2 - V_2h_z^2 + 2pV_4h_r^2 + 2pV_6h_r^2 - 4pV_5h_r^2 = 0 \quad (19.10)$$

430 Let us simplify this equation:

$$V_5(4ph_z^2 + 4ph_r^2) = V_2(2ph_z^2 - h_z^2) + V_8(2ph_z^2 + h_z^2) + 2ph_r^2V_4 + 2ph_r^2V_6 \quad (19.11)$$

431 So we have:

$$V_54p(h_z^2 + h_r^2) = V_2h_z^2(2p - 1) + V_8h_z^2(2p + 1) + V_42ph_r^2 + V_62ph_r^2 \quad (19.12)$$

### 432 19.3 Final forms of relaxation formula

#### 433 19.3.1 zrLV\_RELAX5\_P5\_A

$$h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.13)$$

#### 435 19.3.2 zrLV\_RELAX5\_P5\_B

436 This formula is identical to formula A:

$$h_z \neq h_r$$

$$V_5 = \frac{h_z^2(2p - 1)}{4p(h_z^2 + h_r^2)}V_2 + \frac{h_z^2(2p + 1)}{4p(h_z^2 + h_r^2)}V_8 + \frac{h_r^2}{2(h_z^2 + h_r^2)}(V_4 + V_6) \quad (19.14)$$

#### 438 19.3.3 zrLV\_RELAX5\_P5\_C

$$h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.15)$$

#### 440 19.3.4 zrLV\_RELAX5\_P5\_D

441 This formula is identical to formula C:

$$h_z = h_r = h$$

$$V_5 = \frac{2p - 1}{8p}V_2 + \frac{2p + 1}{8p}V_8 + \frac{1}{4}(V_4 + V_6) \quad (19.16)$$

## 443 20 Relaxation formula for node P6

### 444 20.1 Node description

445 Right edge of mesh ZR.

### 446 20.2 Calculation of relaxation formula

447 Laplace equation at node  $P_6$

$$\nabla^2 (V_{(z,r)})_{P_6} = 0 \quad (20.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} = 0 \quad (20.2)$$

448 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_6$

449

450 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_6} \approx \frac{\frac{V_9 - V_6}{h_r} - \frac{V_6 - V_3}{h_r}}{h_r} = \frac{V_3 + V_9 - 2V_6}{h_r^2} \quad (20.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_6} \approx \frac{1}{r} \frac{V_9 - V_3}{2h_r} = \frac{V_9 - V_3}{2ph_r^2} \quad (20.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_6} \approx \frac{\frac{V_{6z+} - V_6}{h_z} - \frac{V_6 - V_5}{h_z}}{h_z} = \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} \quad (20.5)$$

451 Let us substitute approximations to Laplace equation.

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} + \frac{g_{6z+}}{h_z} = 0 \quad (20.6)$$

$$\frac{V_3 + V_9 - 2V_6}{h_r^2} + \frac{V_9 - V_3}{2ph_r^2} + \frac{V_5 - V_6}{h_z^2} = -\frac{g_{6z+}}{h_z} \quad (20.7)$$

452 Let us find  $V_6$

$$V_6 = ? \quad (20.8)$$

453 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (20.9)$$

454 We obtain

$$2pV_3h_z^2 + 2pV_9h_z^2 - 4pV_6h_z^2 + V_9h_z^2 - V_3h_z^2 + 2pV_5h_r^2 - 2pV_6h_r^2 = -2pg_{6z+}h_zh_r^2 \quad (20.10)$$

455 Let us simplify this equation:

$$V_6(4ph_z^2 + 2ph_r^2) = V_3(2ph_z^2 - h_z^2) + V_9(2ph_z^2 + h_z^2) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.11)$$

456 So we have:

$$V_62p(2h_z^2 + h_r^2) = V_3h_z^2(2p-1) + V_9h_z^2(2p+1) + V_52ph_r^2 + 2pg_{6z+}h_zh_r^2 \quad (20.12)$$

### 457 20.3 Final forms of relaxation formula

#### 458 20.3.1 zrLV\_RELAX5\_P6\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{6z+} &\neq 0 \end{aligned}$$

$$V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 + \frac{g_{6z+}h_zh_r^2}{2h_z^2 + h_r^2} \quad (20.13)$$

#### 460 20.3.2 zrLV\_RELAX5\_P6\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{6z+-} &= 0 \end{aligned}$$

$$V_6 = \frac{h_z^2(2p-1)}{2p(2h_z^2 + h_r^2)}V_3 + \frac{h_z^2(2p+1)}{2p(2h_z^2 + h_r^2)}V_9 + \frac{h_r^2}{2h_z^2 + h_r^2}V_5 \quad (20.14)$$

#### 462 20.3.3 zrLV\_RELAX5\_P6\_C

$$\begin{aligned} h_z &= h_r = h \\ g_{6z+} &\neq 0 \end{aligned}$$

$$V_6 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 + \frac{g_{6z+}h}{3} \quad (20.15)$$

#### 464 20.3.4 zrLV\_RELAX5\_P6\_D

$$\begin{aligned} h_z &= h_r = h \\ g_{6z+} &= 0 \end{aligned}$$

$$V_4 = \frac{2p-1}{6p}V_3 + \frac{2p+1}{6p}V_9 + \frac{1}{3}V_5 \quad (20.16)$$

## 466 21 Relaxation formula for node P7

### 467 21.1 Node description

468 Left, upper corner of mesh ZR.

### 469 21.2 Calculation of relaxation formula

470 Laplace equation at node  $P_7$

$$\nabla^2 (V_{(z,r)})_{P_7} = 0 \quad (21.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} = 0 \quad (21.2)$$

471 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_7$

472

473 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_7} \approx \frac{\frac{V_{7r+} - V_7}{h_r} - \frac{V_7 - V_4}{h_r}}{h_r} = \frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} \quad (21.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_7} \approx \frac{1}{r} \frac{V_{7r+} - V_4}{2h_r} = \frac{V_7 + g_{7r+}h_r - V_4}{2ph_r^2} = \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} \quad (21.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_7} \approx \frac{\frac{V_8 - V_7}{h_z} - \frac{V_7 - V_{7z-}}{h_z}}{h_z} = \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} \quad (21.5)$$

474 Let us substitute approximations to Laplace equation.

$$\frac{V_4 - V_7}{h_r^2} + \frac{g_{7r+}}{h_r} + \frac{V_7 - V_4}{2ph_r^2} + \frac{g_{7r+}}{2ph_r} + \frac{V_8 - V_7}{h_z^2} - \frac{g_{7z-}}{h_z} = 0 \quad (21.6)$$

$$\frac{V_4 - V_7}{h_r^2} + \frac{V_7 - V_4}{2ph_r^2} + \frac{V_8 - V_7}{h_z^2} = -\frac{g_{7r+}}{h_r} - \frac{g_{7r+}}{2ph_r} + \frac{g_{7z-}}{h_z} \quad (21.7)$$

475 Let us find  $V_7$

$$V_7 = ? \quad (21.8)$$

476 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (21.9)$$

477 We obtain

$$\begin{aligned} 2pV_4h_z^2 - 2pV_7h_z^2 + V_7h_z^2 - V_4h_z^2 + 2pV_8h_r^2 - 2pV_7h_r^2 = \\ -2pg_{7r+}h_z^2h_r - g_{7r+}h_z^2h_r + 2pg_{7z-}h_zh_r^2 \end{aligned} \quad (21.10)$$

478 Let us simplify this equation:

$$\begin{aligned} V_7(2ph_z^2 - h_z^2 + 2ph_r^2) = V_4(2ph_z^2 - h_z^2) + V_8(2ph_r^2) + \\ 2pg_{7r+}h_z^2h_r + g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2 \end{aligned} \quad (21.11)$$

479 So we have:

$$\begin{aligned} V_7((2p-1)h_z^2 + 2ph_r^2) = V_4h_z^2(2p-1) + V_82ph_r^2 + \\ 2pg_{7r+}h_z^2h_r + g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2 \end{aligned} \quad (21.12)$$

### 480 21.3 Final forms of relaxation formula

#### 481 21.3.1 zrLV\_RELAX5\_P7\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{7z-} &\neq 0 \\ g_{7r+} &\neq 0 \end{aligned}$$

482

$$\begin{aligned} V_7 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 + \\ \frac{(2p+1)g_{7r+}h_z^2h_r - 2pg_{7z-}h_zh_r^2}{(2p-1)h_z^2 + 2ph_r^2} \end{aligned} \quad (21.13)$$

#### 483 21.3.2 zrLV\_RELAX5\_P7\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{7z-} &= 0 \\ g_{7r+} &= 0 \end{aligned}$$

$$V_7 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2}V_4 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2}V_8 \quad (21.14)$$

484 **21.3.3 zrLV\_RELAX5\_P7\_C**

$$h_z = h_r = h$$

$$g_{7z-} \neq 0$$

$$g_{7r+} \neq 0$$

485

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{7r+} - g_{7z-})}{4p-1} \quad (21.15)$$

486 **21.3.4 zrLV\_RELAX5\_P7\_D**

$$h_z = h_r = h$$

$$g_{7z-} = 0$$

$$g_{7r+} = 0$$

487

$$V_7 = \frac{2p-1}{4p-1}V_4 + \frac{2p}{4p-1}V_8 \quad (21.16)$$

## 488 22 Relaxation formula for node P8

### 489 22.1 Node description

490 Upper edge of mesh ZR.

### 491 22.2 Calculation of relaxation formula

492 Laplace equation at node  $P_8$

$$\nabla^2 (V_{(z,r)})_{P_8} = 0 \quad (22.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} = 0 \quad (22.2)$$

493 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_8$

494

495 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_8} \approx \frac{\frac{V_{8r+} - V_8}{h_r} - \frac{V_8 - V_5}{h_r}}{h_r} = \frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} \quad (22.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_8} \approx \frac{1}{r} \frac{V_{8r+} - V_5}{2h_r} = \frac{V_8 + g_{8r+}h_r - V_5}{2ph_r^2} = \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} \quad (22.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_8} \approx \frac{\frac{V_9 - V_8}{h_z} - \frac{V_8 - V_7}{h_z}}{h_z} = \frac{V_7 + V_9 - 2V_8}{h_z^2} \quad (22.5)$$

496 Let us substitute approximations to Laplace equation.

$$\frac{V_5 - V_8}{h_r^2} + \frac{g_{8r+}}{h_r} + \frac{V_8 - V_5}{2ph_r^2} + \frac{g_{8r+}}{2ph_r} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = 0 \quad (22.6)$$

$$\frac{V_5 - V_8}{h_r^2} + \frac{V_8 - V_5}{2ph_r^2} + \frac{V_7 + V_9 - 2V_8}{h_z^2} = -\frac{g_{8r+}}{h_r} - \frac{g_{8r+}}{2ph_r} \quad (22.7)$$

497 Let us find  $V_8$

$$V_8 = ? \quad (22.8)$$

498 Let us multiply both sides



$$| \cdot 2ph_z^2 h_r^2 \quad (22.9)$$

499 We obtain

$$2pV_5 h_z^2 - 2pV_8 h_z^2 + V_8 h_z^2 - V_5 h_z^2 + 2pV_7 h_r^2 + 2pV_9 h_r^2 - 4pV_8 h_r^2 = \quad (22.10)$$

$$-2pg_{8r+} h_z^2 h_r - g_{8r+} h_z^2 h_r$$

500 Let us simplify this equation:

$$V_8 (2ph_z^2 - h_z^2 + 4ph_r^2) = V_5 (2ph_z^2 - h_z^2) + (V_7 + V_9) 2ph_r^2 + \quad (22.11)$$

$$2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r$$

501 So we have:

$$V_8 ((2p-1)h_z^2 + 4ph_r^2) = V_5 h_z^2 (2p-1) + (V_7 + V_9) 2ph_r^2 + \quad (22.12)$$

$$2pg_{8r+} h_z^2 h_r + g_{8r+} h_z^2 h_r$$

## 502 22.3 Final forms of relaxation formula

### 503 22.3.1 zrLV\_RELAX5\_P8\_A

$$h_z \neq h_r$$

$$g_{8r+} \neq 0$$

$$504 \quad V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) + \quad (22.13)$$

$$\frac{(2p+1)h_z^2 h_r g_{8r+}}{(2p-1)h_z^2 + 4ph_r^2}$$

### 505 22.3.2 zrLV\_RELAX5\_P8\_B

$$h_z \neq h_r$$

$$g_{8r+} = 0$$

$$506 \quad V_8 = \frac{h_z^2 (2p-1)}{(2p-1)h_z^2 + 4ph_r^2} V_5 + \frac{2ph_r^2}{(2p-1)h_z^2 + 4ph_r^2} (V_7 + V_9) \quad (22.14)$$

### 507 22.3.3 zrLV\_RELAX5\_P8\_C

$$h_z = h_r = h$$

$$g_{8r+} \neq 0$$

$$508 \quad V_8 = \frac{2p-1}{6p-1} V_5 + \frac{2p}{6p-1} (V_7 + V_9) + \quad (22.15)$$

$$\frac{(2p+1)h g_{8r+}}{6p-1}$$

509 **22.3.4 zrLV\_RELAX5\_P8\_D**

$$h_z = h_r = h$$

$$g_{8r+} = 0$$

510

$$V_8 = \frac{2p-1}{6p-1}V_5 + \frac{2p}{6p-1}(V_7 + V_9) \quad (22.16)$$

## 511 23 Relaxation formula for node P9

### 512 23.1 Node description

513 Right, upper corner of mesh ZR.

### 514 23.2 Calculation of relaxation formula

515 Laplace equation at node  $P_9$

$$\nabla^2 (V_{(z,r)})_{P_9} = 0 \quad (23.1)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} + \left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} + \left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} = 0 \quad (23.2)$$

516 Approximation of partial derivatives of  $V_{(z,r)}$  at node  $P_9$

517

518 (note:  $r = ph_r$  (such as in Pierre Grivet's book) )

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial r^2} \right)_{P_9} \approx \frac{\frac{V_{9r+} - V_9}{h_r} - \frac{V_9 - V_6}{h_r}}{h_r} = \frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} \quad (23.3)$$

$$\left( \frac{1}{r} \frac{\partial V_{(z,r)}}{\partial r} \right)_{P_9} \approx \frac{1}{r} \frac{V_{9r+} - V_6}{2h_r} = \frac{V_9 + g_{9r+}h_r - V_6}{2ph_r^2} = \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} \quad (23.4)$$

$$\left( \frac{\partial^2 V_{(z,r)}}{\partial z^2} \right)_{P_9} \approx \frac{\frac{V_{9z+} - V_9}{h_z} - \frac{V_9 - V_8}{h_z}}{h_z} = \frac{g_{9z+}}{h_z} + \frac{V_8 - V_9}{h_z^2} \quad (23.5)$$

519 Let us substitute approximations to Laplace equation.

$$\frac{V_6 - V_9}{h_r^2} + \frac{g_{9r+}}{h_r} + \frac{V_9 - V_6}{2ph_r^2} + \frac{g_{9r+}}{2ph_r} + \frac{V_8 - V_9}{h_z^2} + \frac{g_{9z+}}{h_z} = 0 \quad (23.6)$$

$$\frac{V_6 - V_9}{h_r^2} + \frac{V_9 - V_6}{2ph_r^2} + \frac{V_8 - V_9}{h_z^2} = -\frac{g_{9r+}}{h_r} - \frac{g_{9r+}}{2ph_r} - \frac{g_{9z+}}{h_z} \quad (23.7)$$

520 Let us find  $V_9$

$$V_9 = ? \quad (23.8)$$

521 Let us multiply both sides

$$| \cdot 2ph_z^2 h_r^2 \quad (23.9)$$

522 We obtain

$$\begin{aligned} 2pV_6h_z^2 - 2pV_9h_z^2 + V_9h_z^2 - V_6h_z^2 + 2pV_8h_r^2 - 2pV_9h_r^2 = \\ -2pg_{9r+}h_z^2h_r - g_{9r+}h_z^2h_r - 2pg_{9z+}h_zh_r^2 \end{aligned} \quad (23.10)$$

523 Let us simplify this equation:

$$\begin{aligned} V_9(2ph_z^2 - h_z^2 + 2ph_r^2) = V_6(2ph_z^2 - h_z^2) + V_8(2ph_r^2) + \\ 2pg_{9r+}h_z^2h_r + g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2 \end{aligned} \quad (23.11)$$

524 So we have:

$$\begin{aligned} V_9((2p-1)h_z^2 + 2ph_r^2) = V_6h_z^2(2p-1) + V_82ph_r^2 + \\ (2p+1)g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2 \end{aligned} \quad (23.12)$$

### 525 23.3 Final forms of relaxation formula

#### 526 23.3.1 zrLV\_RELAX5\_P9\_A

$$\begin{aligned} h_z &\neq h_r \\ g_{9z-} &\neq 0 \\ g_{9r+} &\neq 0 \end{aligned}$$

527

$$\begin{aligned} V_9 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 + \\ \frac{(2p+1)g_{9r+}h_z^2h_r + 2pg_{9z+}h_zh_r^2}{(2p-1)h_z^2 + 2ph_r^2} \end{aligned} \quad (23.13)$$

#### 528 23.3.2 zrLV\_RELAX5\_P9\_B

$$\begin{aligned} h_z &\neq h_r \\ g_{9z-} &= 0 \\ g_{9r+} &= 0 \end{aligned}$$

$$V_9 = \frac{h_z^2(2p-1)}{(2p-1)h_z^2 + 2ph_r^2} V_6 + \frac{2ph_r^2}{(2p-1)h_z^2 + 2ph_r^2} V_8 \quad (23.14)$$

### 529 23.3.3 zrLV\_RELAX5\_P9\_C

$$h_z = h_r = h$$

$$g_{9z-} \neq 0$$

$$g_{9r+} \neq 0$$

530

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 + \frac{h((2p+1)g_{9r+} + g_{9z+})}{4p-1} \quad (23.15)$$

### 531 23.3.4 zrLV\_RELAX5\_P9\_D

$$h_z = h_r = h$$

$$g_{9z-} = 0$$

$$g_{9r+} = 0$$

532

$$V_9 = \frac{2p-1}{4p-1}V_6 + \frac{2p}{4p-1}V_8 \quad (23.16)$$

## 533 References

- 534 [1] P. Grivet, *Electron Optics, Second (revised) English edition*. Pergamon  
535 Press Ltd., 1972.
- 536 [2] J. R. Nagel, "Solving the generalized poisson equation using the fi-  
537 nite - difference method (fdm).". [https://my.ece.utah.edu/~ece6340/](https://my.ece.utah.edu/~ece6340/LECTURES/Feb1/,)  
538 LECTURES/Feb1/, 2012. [Online; accessed 2-February-2023].
- 539 [3] kryomaxim, "taylor expansion in cylindrical coordinates."  
540 [https://math.stackexchange.com/questions/1133311/](https://math.stackexchange.com/questions/1133311/taylor-expansion-in-cylindrical-coordinates)  
541 taylor-expansion-in-cylindrical-coordinates, 2015. [Online;  
542 accessed 9-May-2023].
- 543 [4] A. Septier(ed.), *Focusing of Charged Paticles. Volume I*. New York and  
544 London, Academic Press, 1967.
- 545 [5] A. Septier(ed.), *Applied Charged Paticle Optics, part A*. New York and Lon-  
546 don, Academic Press, 1980.
- 547 [6] D. W. O. Heddle, *Electrostatic Lens Systems. Second Edition*. Institute of  
548 Physics Publishing, Bristol and Philadelphia, 2000.
- 549 [7] B. Paszkowski, *Optyka Elektronowa, wydanie II, poprawione i uzupełnione*.  
550 Państwowe Wydawnictwa Naukowo - Techniczne, Warszawa, 1965.
- 551 [8] B. Paszkowski, *Electron Optics [by] B.Paszkowski. Translated from the Pol-  
552 ish by George Lepa. English translation edited by R. C. G. Leckey*. London,  
553 Iliffe; New York, American Elsevier Publishing Company Inc., 1968.